Systems of differential equations - the linear case

Recall that a single linear differential equation

$$\frac{d}{dt}x(t) = ax(t), \qquad x(0) = x_0$$

has the solution $x(t) = e^{at}x_0$. In particular, if $a > 0$, the solution grows to infinity, and if $a < 0$ then the solution approaches zero. We now want to generalize this result to two coupled equations

$$\begin{aligned} \frac{d}{dt}x_1(t) &= ax_1(t) + bx_2(t), & x_1(0) = x_{10}, \\ \frac{d}{dt}x_2(t) &= rx_1(t) + sx_2(t), & x_2(0) = x_{20}. \end{aligned}$$

We can also write this in matrix notation (suppressing the argument t for the moment) as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ r & s \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =: A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ r & s \end{bmatrix}.$$

Observation I: Eigenvalues and eigenvectors provide solutions

Suppose we are looking for solutions of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Plugging this expression into the equation above, we get

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \lambda e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = e^{\lambda t}A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Canceling the exponential term on both sides, we find the condition

$$\lambda\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

which means that λ has to be an eigenvalue of A and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ has to be the corresponding eigenvector.

Fact: If λ is an eigenvalue of A and v is the corresponding eigenvector, then

$$x(t) = e^{\lambda t}v$$

is a solution of the linear system of differential equations

$$\frac{d}{dt}x(t) = Ax(t),$$

where $x = (x_1, \dots x_n)^T$, and A is an $n \times n$-matrix.

Example 1

Take the system

$$\begin{aligned}\frac{d}{dt}x_1(t) &= x_1(t) + 4x_2(t),\\ \frac{d}{dt}x_2(t) &= 2x_1(t) - x_2(t),\end{aligned}$$

with matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$. The eigenvalues of A are given by the equation

$$(1-\lambda)(-1-\lambda) - 8 = \lambda^2 - 9 = 0.$$

Hence, the eigenvalues are $\lambda = 3$ and $\mu = -3$. The corresponding eigenvectors are

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

respectively. Hence, we have the two solutions

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\lambda t}v = e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\mu t}w = e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have two solutions, but we don't have any constants in the equation yet with which we could match the initial condition. This is the next topic.

Observation II: Sums and multiples of solutions are solutions

Suppose we have two solutions $w(t)$ and $z(t)$ for the same system, i.e.,

$$\frac{d}{dt}w(t) = Aw(t), \qquad \frac{d}{dt}z(t) = Az(t).$$

Now pick two numbers C_1, C_2 and form

$$x(t) = C_1 w(t) + C_2 z(t).$$

Then $x(t)$ is also a solution since

$$\frac{d}{dt}x(t) = C_1\frac{d}{dt}w(t) + C_2\frac{d}{dt}z(t) = C_1 Aw(t) + C_2 Az(t) = A[C_1 w(t) + C_2 z(t)] = Ax(t).$$

Example 1, continued

Find the solution of the system

$$\begin{aligned}\frac{d}{dt}x_1(t) &= x_1(t) + 4x_2(t),\\ \frac{d}{dt}x_2(t) &= 2x_1(t) - x_2(t),\end{aligned}$$

with initial values $x_{10} = 3, x_{20} = 3$.
We already know the eigenvalues and eigenvectors, and hence, we get the general solution as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Substituting the initial conditions, i.e., setting $t = 0$, we get

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence we solve the system

$$\begin{bmatrix} 2 & 1 & | & 3 \\ 1 & -1 & | & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & | & 3 \\ 0 & 3 & | & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -1 \end{bmatrix}.$$

The solution is $C_1 = 2, C_2 = -1$, and therefore the solution to the differential equation is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

or, equivalently

$$\begin{aligned} x_1(t) &= 4e^{3t} - e^{-3t}, \\ x_2(t) &= 2e^{3t} + e^{-3t}. \end{aligned}$$

We can always check our solution by differentiating:

$$\frac{d}{dt} x_1(t) = 12e^{3t} + 3e^{-3t}, \quad x_1 + 4x_2 = 4e^{3t} - e^{-3t} + 8e^{3t} + 4e^{-3t} = 12e^{3t} + 3e^{-3t}.$$

The two expressions agree. Similar for x_2.

Example 2

Find the solution of Newton's law of cooling

$$\begin{aligned} \frac{d}{dt} x_1(t) &= 3[x_2(t) - x_1(t)], \qquad x_{10} = 5 \\ \frac{d}{dt} x_2(t) &= x_1(t) - x_2(t), \qquad x_{20} = 1. \end{aligned}$$

First, we observe that the eigenvalues of the matrix $A = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$ are $\lambda = -4, \mu = 0$, which are distinct real numbers. The corresponding eigenvectors are

$$v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence, the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 e^{-4t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The constants are given from the linear system

$$\left[\begin{array}{cc|c} 3 & 1 & 5 \\ -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right].$$

Hence the solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-4t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Explicit solutions in the case of distinct real eigenvalues: To solve the system

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

do the following

1. Find the eigenvalues λ, μ of A.
2. If $\lambda \neq \mu$ are real numbers, then find the corresponding eigenvectors v, w.
3. Find the constants C_1, C_2 such that
$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = C_1 v + C_2 w.$$
4. The solution is
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda t} v + C_2 e^{\mu t} w.$$

If the eigenvalues λ, μ are not real numbers or if $\lambda = \mu$, then the procedure is similar but a bit more tricky. We will not consider these cases here.

Example 3

Find the solution of the system

$$\begin{aligned} \frac{d}{dt}x_1(t) &= x_1(t) + 4x_2(t), \qquad x_{10} = 4 \\ \frac{d}{dt}x_2(t) &= x_1(t) + x_2(t), \qquad x_{20} = 8. \end{aligned}$$

First, we observe that the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ are $\lambda = 3, \mu = -1$, which are distinct real numbers. The corresponding eigenvectors are

$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Hence, the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

The constants are given from the linear system

$$\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 1 & -1 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right].$$

Hence the solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{3t} \begin{bmatrix} 10 \\ 5 \end{bmatrix} - e^{-t} \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

Observation III: Stability of zero

Consider the system of differential equations

$$\frac{d}{dt} x(t) = Ax(t), \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

and assume that λ is an eigenvalue of A with corresponding eigenvector v. Then we know that

$$x(t) = e^{\lambda t} v$$

is a solution. If λ is a real number and $\lambda > 0$, then this solution grows in time, but if $\lambda < 0$ then the solution decays to zero. If $\lambda = a + b\mathrm{i}$ is a complex number, then we use Euler's formula

$$e^{\lambda t} = e^{(a+b\mathrm{i})t} = e^{at}[\cos(bt) + \mathrm{i}\sin(bt)].$$

We see that the solution grows if the real part Re(λ)=$a > 0$, and the solution decays to zero if the real part Re(λ)=$a < 0$.

Fact: If all the eigenvalues of the matrix A have negative real part, then all the solutions to the system of differential equations

$$\frac{d}{dt} x(t) = Ax(t), \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

decay to zero as $t \to \infty$. If one eigenvalue has positive real part, then there is at least one solutions that does not decay to zero. (In fact, there is a solution that grows to infinity in an appropriate norm).
Note that we make no statement about the case of eigenvalues with zero real part.

Examples revisited

1. In the first example above, the matrix
$$A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$
has the eigenvalues $\lambda = 3 > 0$ and $\mu = -3 < 0$. Hence there is one solution that grows to infinity, namely
$$x(t) = e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

2. In Newton's law of cooling, we have the matrix
$$A = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$$
with eigenvalues $\lambda = -4, < 0$ and $\mu = 0$. Hence, in this case the box above does not apply. We see that we have the constant solution
$$x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
that neither decays to zero nor grows to infinity.

3. In the third example above, we had one positive and one negative eigenvalue, hence there is one solution that grows to infinity.

Example 4

Consider the matrix
$$A = \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix}.$$
The eigenvalues are $\lambda = -1, \mu = -3$. Both are real and negative, hence all solutions of
$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
converge to zero.

Example 5

Consider the matrix
$$A = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}.$$
The eigenvalues are $\lambda = -1 + 4\mathrm{i}, \mu = -1 - 4\mathrm{i}$. Both have negative real part, hence all solutions of
$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
converge to zero.

Practice problems

I) Find the explicit solution of the linear system of differential equations together with initial conditions.

1.
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$

2.
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

3.
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

II) In the following cases, determine whether all solutions of the system $\frac{d}{dt}x = Ax$ converge to zero or not. [Hint: find the real parts of the eigenvalues of A.]

(a) $\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$, (c) $\begin{bmatrix} -1 & 4 \\ -3 & 7 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, (e) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$, (f) $\begin{bmatrix} -2 & 5/2 \\ -1/2 & 0 \end{bmatrix}$,

Solutions to practice problems

I) Explicit solitions

1. The eigenvalues of the matrix are $\lambda = 3$ and $\mu = -1$. The corresponding eigenvectors are
$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$
The initial condition gives the linear system
$$C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$
which has the solution $C_1 = 3, C_2 = -2$. Hence the solution is
$$\begin{bmatrix} x \\ y \end{bmatrix} = 3e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2e^{-t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

2. The eigenvalues of the matrix are $\lambda = 3$ and $\mu = 2$. The corresponding eigenvectors are
$$v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
The initial condition gives the linear system
$$C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
which has the solution $C_1 = 1, C_2 = -3$. Hence the solution is
$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} - 3e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

3. The eigenvalues of the matrix are $\lambda = -6$ and $\mu = -2$. The corresponding eigenvectors are
$$v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
The initial condition gives the linear system
$$C_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$
which has the solution $C_1 = -1, C_2 = 2$. Hence the solution is
$$\begin{bmatrix} x \\ y \end{bmatrix} = -e^{-6t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + 2e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

II) Stability

(a) The eigenvalues are $\lambda = 3 + 2\mathrm{i}$ and $\mu = 3 - 2\mathrm{i}$. Both have positive real part, hence no solution except for zero converges to zero.

(b) The eigenvalues are $\lambda = 3, \mu = -3$. One is real and positive, hence there is a solution that does not converge to zero.

(c) The eigenvalues are $\lambda = 5, \mu = 1$. Both are real and positive, hence all solutions (except zero) grow to infinity.

(d) The eigenvalues are $\lambda = 1+2\mathrm{i}, \mu = 1-2\mathrm{i}$. Both have positive real part, hence all solutions (except zero) grow to infinity.

(e) The eigenvalues are $\lambda = \sqrt{34}, \mu = -\sqrt{34}$. One is real and positive, hence there is a solution that does not converge to zero.

(e) The eigenvalues are $\lambda = -1 + \mathrm{i}/2, \mu = -1 - \mathrm{i}/2$. Both have negative real part, hence all solutions converge to zero.

Systems of differential equations - the nonlinear case

There are some similarities and many differences between single differential equations and systems of differential equations and how they are treated. Here we fist give a general recipe for how to study systems of two equations, and then we summarize similarities and differences in a table.

A system of two differential equations can be written in the form

$$\begin{aligned}\frac{d}{dt}x &= F(x,y),\\ \frac{d}{dt}y &= G(x,y),\end{aligned}$$

where $x = x(t), y = y(t)$ are the two functions that we are looking for. In general, there is no explicit solution available, i.e., the functions $x(t)$ and $y(t)$ cannot be written down. Nonetheless, we can find out the general shape of solutions. This is done in two parts:

1. Phase plane (as explained in the textbook in chapters 5.6-5.8)
2. Linear stability analysis (not in the textbook, but mostly in the previous section of the lecture notes)

Definitions

The x-**nullcline** is the set of all points (x,y) where $x(t)$ does not change, i.e., $dx/dt = 0$, i.e., $F(x,y) = 0$.
The y-**nullcline** is the set of all points (x,y) where $y(t)$ does not change, i.e., $dy/dt = 0$, i.e., $G(x,y) = 0$.
A **steady state** or **equilibrium** is a point where neither x nor y change, i.e., $dx/dt = 0$ AND $dy/dt = 0$ or, equivalently, $F(x,y) = 0$ AND $G(x,y) = 0$.
A **direction arrow** is a vector that indicates in which direction the solution will go from a given point. The direction arrow at the point (x,y) has the coordinates $(F(x,y), G(x,y))$.
The **linearzation** of the system at a steady state (x^*,y^*) is given by the linear system

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = J(x^*,y^*)\begin{bmatrix} X \\ Y \end{bmatrix}, \qquad J(x,y) = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix},$$

where $X(t), Y(t)$ are the small deviations from x^*, y^*, i.e., $x(t) = x^* + X(t), y(t) = y^* + Y(T)$.

Recipe

Given the system

$$\begin{aligned} \frac{d}{dt}x &= F(x,y), \\ \frac{d}{dt}y &= G(x,y), \end{aligned}$$

do the following steps.

1. Phase-Plane
   (a) Find the nullcline of x, i.e., the set of points where $dx/dt = 0$, or equivalently $F(x,y) = 0$.
   (b) Find the nullcline of y, i.e., the set of points where $dy/dt = 0$, or equivalently $G(x,y) = 0$.
   (c) Draw these two sets in the x-y-plane.
   (d) Find the steady states, i.e., the points where $F(x,y) = 0$ AND $G(x,y) = 0$, i.e., the intersection of the nullclines.
   (e) On each of the nullclines, draw the direction arrows.
   (f) In each of the regions in space in between the nullclines, draw the direction arrows.
2. Linear stability analysis
   (a) Find the Jacobi matrix
   $$J(x,y) = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}.$$
   (b) Evaluate the Jacobi matrix at each stationary point and find the real parts of the eigenvalues.
   (c) If the real parts are negative, then the stationary point is stable; if one real part is positive, it is unstable.
3. For a given initial condition, draw a solution into the phase plane. Then plot the two components of the solution as a function of time.

Table 1: Summary and comparison of one- and two-dimensional differential equations and the corresponding solution techniques.

	1-D	2-D
Linear equation	$\frac{dx}{dt} = rx$	$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ $A = \begin{bmatrix} a & b \\ c & e \end{bmatrix}$
Solution to linear equation	$x(t) = e^{rt}x_0$	$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda t} v + C_2 e^{\mu t} w$ $Av = \lambda v,\ Aw = \mu w$
Nonlinear equation	$\frac{dx}{dt} = f(x)$	$dx/dt = F(x, y)$ $dy/dt = G(x, y)$
Explicit solution	Separation of variables $\int \frac{dx}{f(x)} = \int dt$	Generally not available
Graphical Solution	Phase-line diagram Slope field	Phase-plane
Steady state	$f(x^*) = 0$	$F(x^*, y^*) = 0, G(x^*, y^*) = 0$
Stability	$f'(x^*) < 0$	Eigenvalues of $J(x^*, y^*)$ have negative real part $J = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}$